① Integration on $\mathbb{R}^n$

f cpt supp on $\mathbb{R}^n$, or on a closed subdomain D

- Fubini + iterated $\int$

$$\int_D f\,dv = \int \cdots \left(\int f\,dx^1\right)\cdots dx^n$$

Fubini's thm – order doesn't matter

- Change-of-variables formula

$D' = G(D)$, $f = f'\circ G$

$$\int_{D'} f'\,dv' = \int_D \underbrace{f'(G(x))}_{f} \left|\det \frac{\partial G^i}{\partial x^j}\right| dv$$

↑ change in vol of

on mfld

$\int_M f\,dx$ cannot make sense but $\int_M f\,dv$ can make sense for a Riem. mfld:

Thm If M Riem, f on M, the following is indep of choices:

η_α POU sub to charts $U_\alpha \xrightarrow{\psi_\alpha} M$

$$\sum_\alpha \int_{U_\alpha} \eta_\alpha f \sqrt{\det g_{ij}}$$

[• refine partitions
• change of var's]

This is $\int_{(M,g)} f\,dv$

Q: Is $f\sqrt{\det g_{ij}}$ a component of a tensor?

What about FTC $\int_a^b \frac{\partial F}{\partial x} = F(b) - F(a)$ ...... Friday

As it stands, only holds if $a < b$ ← seems ... but actually important.

Eg.: If $a<b$

$$\int_b^a f = \int_a^b f$$

~~breaks~~ nice rule $\int_a^b + \int_b^c = \int_a^c$

Want: A different kind of integration that remembers this.

② Orientation

<u>Defn</u>: section of $\Lambda^{top} M - 0/\mathbb{R}^+$ ← or. cover conn or discon.

- Standard or of $\mathbb{R}^n$ (<u>std</u> or of $\mathbb{R}^0$ is a + or −)
- Nonexamples

  in particular, Riem metric $\not\Rightarrow$ or.

  Now have or Riem mfd has a canonical top form

  $$\Omega(e_1, \dots, e_n) = \pm\sqrt{\det(g_{ij})}$$

  ↑ sel. in right component

  (check well defined: $\Omega(Ae_1, \dots, Ae_n) = \det A \, \Omega(e_1, \dots, e_n)$

  $$\det(g(Ae_i, Ae_j)) = \det(A^T g A) = |\det A| \sqrt{\det g}$$

- local diffeo between or. M.'s or. pres or or-rever
- compatible charts → or.

  ↑ section on U_1, U_2 agreeing on $U_1 \cap U_2$ → section on $U_1 \cup U_2$

③ Integration of top-forms

n-form on $\mathbb{R}^n$ w/ cpt support

$\omega = f\, dx^1 \wedge \cdots \wedge dx^n \longrightarrow \int_{\mathbb{R}^n, \text{std}} \omega := \int_{\mathbb{R}^n} f$

Then if ω section of $\Lambda^n M$, $\underline{M \text{ oriented}}$, can define

$$\int_M \omega = \sum_{\text{charts}} \text{or}(\varphi) \int \varphi^*(\eta_\alpha \omega)$$

orient should be part of the notation!

↑ in domain ≥ 1, $\sum_{\text{or charts}} \int \varphi^*(\eta_\alpha \omega)$

Properties

- linear on $\Lambda^n_0(M)$
- $\int_{-M} \omega = - \int_M \omega$
- $\int_M F^* \omega = \text{or}(F) \int_{M'} \omega$ if $F: M \to M'$ diffeo

A: what is the thing above?

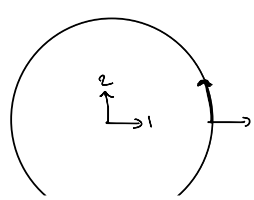
Fiber bundle $|\Lambda^n M|$ <u>not</u> a vector bundle.

should exclude 0 ⇝ it is a fiber bundle.

Q: what about domains?

④ Mflds-w-∂

- Defn
- $\partial^2 = 0$
- Compact support
- Boundary orientation
  - outward normal (1D)
  - put it first (2D)



FTOC $\quad \int_I df = \int_{\partial I} f$

$$\int_H d\omega = \int_{\partial H} \omega$$

To generalize this, define

(5) Exterior derivative

(6) Stokes thm